Prop 2.7 Let $f:(X,d)\to(Y,\rho)$ be a mapping between metric spaces.

(a) f is continuous at x
$\Leftrightarrow$ $\forall$ open set G (in Y) containing $f(x)$,

$f^{-1}(G)$ contains $B_\varepsilon(x)$ for some $\varepsilon>0$.

(b) f is continuous in X
$\Leftrightarrow$ $\forall$ open set G in Y, $f^{-1}(G)$ is open in X

Pf: (a) ($\Rightarrow$) Suppose not,

then $\exists$ open set G in Y containing $f(x)$

s.t. $f^{-1}(G)$ doesn't containing $B_\varepsilon(x)$, $\forall\varepsilon>0$.

i.e. $B_\varepsilon(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall\varepsilon>0$.

In particular $B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]\neq\phi$, $\forall n$.

Pick $x_n\in B_{\frac{1}{n}}(x)\cap[X\setminus f^{-1}(G)]$, $\forall n$.

Then
$$\begin{cases} x_n\in B_{\frac{1}{n}}(x) \Rightarrow x_n\to x \text{ as } n\to\infty \\ x_n\in X\setminus f^{-1}(G)\Rightarrow f(x_n)\notin G, \forall n \end{cases}$$

By Prop 2.5, $f(x_n)\not\to f(x)$. Contradicting the assumption that f is cts. at x.

($\Leftarrow$) $\forall \varepsilon > 0$, $B_\varepsilon(f(x)) \subset Y$ is an open set containing $f(x)$. By assumption,

$$f^{-1}(B_\varepsilon(f(x))) \supset B_\delta(x) \text{ for some } \delta,$$

i.e. $f(y) \in B_\varepsilon(f(x))$, $\forall y \in B_\delta(x)$

$\Rightarrow$ $d(f(y), f(x)) < \varepsilon$, $\forall d(y,x) < \delta$.

$\therefore$ f is cts. at x.

(b) follows from (a). (Ex!) $\divideontimes$

<u>Note</u>: We also have:

f is cts in X

$\Longleftrightarrow$ $\forall$ closed set $F \subset Y$, $f^{-1}(F)$ is closed in X.

(Pf: Ex!)

<u>Eg</u>: (i) Let $A \subset X$ & $A \neq \emptyset$.

Since $\rho_A(x) = d(x, A)$ is cts,

in $\mathbb{R}$

$$G_r = \{x \in X : d(x, A) < r\} = \rho_A^{-1}(B_r(0))$$

is open in X.

(ii) <u>Claim</u>: If A is closed ($\& \neq \emptyset$), then $A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$.

Hence any closed set is a <u>countable intersection</u> of open sets.

Pf: It is clear that $A \subset \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ as $A \subset G_{\frac{1}{n}}$, $\forall n$.

Let $x \in \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ then $x \in G_{\frac{1}{n}}$, $\forall n$

$\Rightarrow \quad d(x, A) < \frac{1}{n}, \quad \forall n$

$\Rightarrow \quad \exists\, x_n \in A$ s.t. $d(x, x_n) < \frac{1}{n}, \quad \forall n$

Hence $\{x_n\} \subset A$ is a seq in A s.t. $x_n \to x$.

Since A is closed, we have $x \in A$.

$\therefore \quad A = \bigcap_{n=1}^{\infty} G_{\frac{1}{n}}$ ※

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§2.3 <u>Points in Metric Spaces</u>

<u>Def</u>: Let E be a set in a metric space (X, d)

(1) A point $x \in X$ (not nec. in E) is called a <u>boundary point</u> of E if $\forall$ open set $G \subset X$ containing x, $G \cap E \neq \emptyset$ & $G \setminus E \neq \emptyset$
$(G \cap (X \setminus E) \neq \emptyset)$

Note: It suffices to check G of the form $B_\varepsilon(x)$ for all small $\varepsilon>0$, or even $B_{\frac{1}{n}}(x)$, $\forall n\geq 1$

(2) The set of boundary points of E will be denoted by ∂E and is called the boundary of E.

(3) The closure of E, denoted by $\overline{E}$, is defined to be $\overline{E} = E \cup \partial E$.

Note: $\partial E = \partial(X\setminus E)$, $\forall E \subset X$.

eg: For $B_r(x) = \{y\in X : d(y,x)<r\}$ in $(\mathbb{R}^n$, standard$)$

$$\partial B_r(x) = S_r(x) = \{y\in X : d(y,x)=r\} \quad \&$$

$$\overline{B_r}(x) = B_r(x)\cup \partial B_r(x)$$
$$= \{y\in X : d(y,x)\leq r\}.$$

Notes (i) $\partial\emptyset = \emptyset$ (Ex!)

(ii) $\forall E\subset X$, ∂E is a closed set.

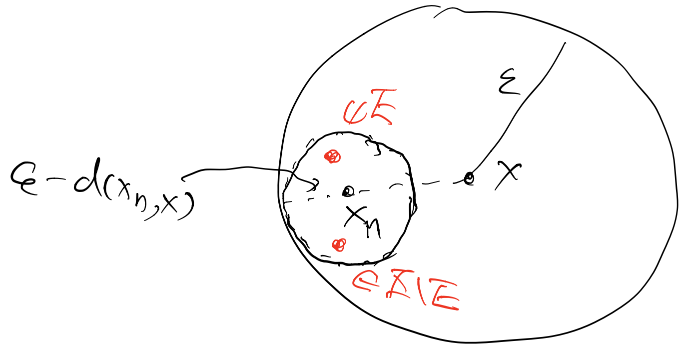
Pf of (ii): Consider a seq $\{x_n\}\subset\partial E$ converging to some $x\in X$.

Then $\forall \varepsilon>0$, $x_n\in B_\varepsilon(x)$ for $n\geq n_0$ (for some n_0)

$\Rightarrow B_{\varepsilon-d(x_n,x)}(x_n)\subset B_\varepsilon(x)$.

As $x_n\in\partial E$,

$$\begin{cases} B_{\varepsilon - d(x_n,x)}(x_n) \cap E \neq \phi \\ B_{\varepsilon - d(x_n,x)}(x_n) \setminus E \neq \phi \end{cases}$$



$$\Rightarrow \begin{cases} B_\varepsilon(x) \cap E \neq \phi \\ B_\varepsilon(X) \setminus E \neq \phi \end{cases}$$

$\Rightarrow \quad x \in \partial E$. Therefore ∂E is closed ※

(iii) $\forall\ E \subset X$, $\overline{E}$ is a closed set.

<u>Pf</u>: Let $x \in X \setminus \overline{E}$,

then $x \notin \partial E$ (& $x \notin E$)

$\Rightarrow$ $\exists$ open set $B_\varepsilon(x)$ s.t.

either $B_\varepsilon(x) \cap E = \phi$ or $B_\varepsilon(x) \setminus E = \phi$

As $x \notin E$, $x \in B_\varepsilon(x) \setminus E \Rightarrow B_\varepsilon(x) \setminus E \neq \phi$.

So we must have $B_\varepsilon(X) \cap E = \phi$

$\Rightarrow$ $B_\varepsilon(x) \subset X \setminus E$.

<u>Claim</u> $B_{\frac{\varepsilon}{2}}(x) \subset X \setminus \overline{E}$.

<u>Pf of Claim</u>: Clearly $B_{\frac{\varepsilon}{2}}(x) \subset X \setminus E$.

Suppose $B_{\frac{\varepsilon}{2}}(x) \cap \partial E \neq \emptyset$.

Then $\exists\, y \in B_{\frac{\varepsilon}{2}}(x) \cap \partial E$

$$y \in \partial E \Rightarrow B_{\frac{\varepsilon}{2}}(y) \cap E \neq \emptyset$$

$$\Rightarrow \exists\, z \in B_{\frac{\varepsilon}{2}}(y) \cap E.$$

Note that $B_{\varepsilon}(x) \subset X \setminus E$ & $z \in E$

$$\Rightarrow \quad d(x,z) > \varepsilon$$

$$\therefore \quad \varepsilon < d(x,z) \leq d(x,y) + d(y,z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Which is a contradiction.

$\therefore$ $B_{\varepsilon/2}(x) \cap \partial E = \emptyset$ & hence

$B_{\varepsilon/2}(x) \subset X \setminus E$. (completed the proof of Claim.)

Finally, the claim shows that $X \setminus \overline{E}$ is open & hence $\overline{E}$ is closed. ⊠

<u>Prop 2.8</u> Let E be a set in a metric space (X, d). We have

$$\overline{E} = \cap \{ C : C = \text{closed set}, C \supset E \}.$$

<u>Pf</u>: By Note (iii), $\overline{E}$ is closed & $\overline{E} \supset E$

$$\therefore \overline{E} \in \{ C : C = \text{closed set}, C \supset E \}$$

$$\Rightarrow \overline{E} \supset \cap \{ C : C = \text{closed set}, C \supset E \}$$

Conversely, let C be a closed set & $C \supset E$.

Then $\forall\, y \in \partial E$, $\begin{cases} B_{\frac{1}{n}}(y) \cap E \neq \phi \\ B_{\frac{1}{n}}(y) \setminus E \neq \phi \end{cases}$, $\forall n$

$\Rightarrow \exists\, y_n \in B_{\frac{1}{n}}(y) \cap E$, $\forall n$

Then $\begin{cases} y_n \in E \subset C \quad \& \\ y_n \to y \quad \text{as } n \to +\infty \end{cases}$

$\Rightarrow y \in C$ (as C is closed)

$\therefore \quad \partial E \subset C \Rightarrow \overline{E} \subset C$

$\therefore \quad \overline{E} \subset \bigcap \{C : C = \text{closed}, C \supset E\}$ $\#$

<u>Def</u>: Let E be a subset of a metric space (X, d).

(1) A point x is called an <u>interior point</u> of E if $\exists$ an open set G s.t. $x \in G$ & $G \subset E$.

(2) The set of all interior points of E is called the <u>interior of E</u>, denoted by E^{o}.

<u>Notes</u>: (i) E^{o} is open

(ii) $E^{o} = E \setminus \partial E$

(iii) $E^{o} = X \setminus \overline{(X \setminus E)}$

(iv) $E^{o} = \bigcup \{G : G = \text{open} \;\&\; G \subset E\}$

(Pf: Ex!)

eg 2.16 $E = \mathbb{Q} \cap [0,1]$ in $(X = [0,1],\ d(x,y) = |x-y|)$

Then $E^\circ = \emptyset$ & $\overline{E} = [0,1]$, $\partial E = ?$

eg 2.17 Let D be a domain in $\mathbb{R}^2$ bounded by several cts. curves S.

Then $\partial D = S$

$\overline{D} = D \cup S = D \cup \partial D$

& $(\overline{D})^\circ = D$.

eg 2.18 : (i) $\overline{E \cup F} = \overline{E} \cup \overline{F}$ for $E, F \subset (X, d)$

(Ex!)

(ii) However $(E \cup F)^\circ \neq E^\circ \cup F^\circ$ in general.

Counterexample: $E = \mathbb{Q}$, $F = \mathbb{R} \setminus \mathbb{Q}$ $\quad (X, d) = (\mathbb{R}, \text{standard})$

Then $E \cup F = \mathbb{R}$

$\Rightarrow (E \cup F)^\circ = \mathbb{R}$

However $E^\circ = F^\circ = \emptyset$

$\Rightarrow E^\circ \cup F^\circ = \emptyset \neq \mathbb{R} = (E \cup F)^\circ$.

(iii) We only have $E^\circ \cup F^\circ \subset (E \cup F)^\circ$ (Pf: Ex!)

eg 2.19 : $(X = C[0,1],\ d_\infty(f,g) = \|f-g\|_\infty)$

Let $S = \{ f \in X : 1 < f(x) \leq 5,\ \forall x \in [0,1] \}$

(1) <u>Claim</u>: $\overline{S} = \{f \in X : 1 \le f(x) \le 5, \ \forall x \in [0,1]\}$.

Pf: Let $C = \{f \in X : 1 \le f(x) \le 5, \ \forall x \in [0,1]\}$.

Then $C = \{1 \le f(x)\} \cap \{f(x) \le 5\}$

closed in (X, d_∞)

$\therefore$ C is closed.

$\therefore$ $\overline{S} \subset C$

Conversely, $\forall f \in C$

$$f_n(x) = \max\{f(x), 1 + \tfrac{1}{n}\} \in X = C[0,1].$$

$\forall n$.

Then

$$1 < 1 + \tfrac{1}{n} \le f_n(x) \le 5, \quad \forall n$$

$\Rightarrow$ $f_n(x) \in S$.

Note $d_\infty(f_n, f) = \max_{x \in [0,1]} |f_n - f|(x)$

$$\le 1 + \tfrac{1}{n} - 1 = \tfrac{1}{n} \to 0 \text{ as } n \to \infty$$

$\therefore$ $f \in \overline{S}$ as $f_n \to f$.

Hence $C \subset \overline{S}$. ※

(2) <u>Claim</u>: $S^0 = \{f \in X : 1 < f(x) < 5, \ \forall x \in [0,1]\}$.

(Pf: Ex!)